Numerical solutions of a class of nonlinear fractional boundary value problems with beta derivative

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Abstract

In this paper, a class of fractional order nonlinear differential equations with boundary conditions is studied. We employ Sinc-Galerkin method to obtain the approximate solution for investigated fractional boundary value problems. A local derivative so called beta-derivative definition is preferred for fractional order. In order to illustrate the efficiency of the proposed method on investigated class of functions, two examples with numerical simulations are tested. The error estimation shows that Sinc-Galerkin method is a consistent and effective method.

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1 Introduction

In recent years numerical solutions of fractional differential equations with local derivative definitions gain popularity in the literature. That is mostly due to some advantages of local derivatives such as satisfying many properties which causes limitations for the fractional derivatives having memory effect. Some articles have been focused on analytical or numerical solutions of the fractional differential equations involving beta-derivatives. In [8] authors obtained new optical soliton solutions for CLL equation with beta derivative in optical fibers. The exact analytical solutions of nonlinear Schr odinger equation (NLSE) with the beta-derivative are constructed in [9] by using sub-equation method. In [10], the authors focus on exact solutions of a biological population model namely, the DNA Peyrard-Bishop dynamic equation involving beta-derivative by using three different methods. More applications of beta-derivative can be found in [2, 3, 5, 6, 11, 12].
In this study, we use the sinc-Galerkin method that has almost not been developed for the class of fractional nonlinear differential equations in the form

$$L[y(x)] = y''(x) + p(x)y'(x) + q(x)y(x) + n(x)y^s y^{(\beta)} = f(x), 0 < \beta \leq 1 \tag{1.1}$$

subject to homogeneous boundary conditions

$$y(a) = 0, y(b) = 0 \tag{1.2}$$

where *beta* denotes beta derivative defined in [1]. We refer interested reader to [14, 15, 16] and references therein for details on sinc-Galerkin method.

The structure of the paper is organized as follows. In the following section some basic definitions are given. In the third section main theorems for the function approximation to (1.1) based on the sinc-Galerkin method are given. Then in Section 4 error estimation is given. In Section 5 proposed method is illustrated with numerical examples. Finally, in the last section the paper is concluded.

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2 Preliminaries and notation

In this section, some preliminaries and notations related to sinc basic functions and beta-derivative are given. The definition of beta-derivative is as follows:

Definition 2.1. Let f be a function, such that, $f : [a, \infty) \to \mathbb{R}$. Then the beta-derivative is defined as:

$$D_x^\beta(f(x)) = \lim_{\varepsilon \to 0} \frac{f\Big(x + \varepsilon\big(x + \frac{1}{\Gamma(\beta)}\big)^{1-\beta}\Big) - f(x)}{\varepsilon}$$

for all $x \geq a, \beta$ in $(0, 1]$. Then is the limit of the above exists, f is says to be beta-differentiable.

Theorem 2.2. Asuming that $f : [a, \infty) \to \mathbb{R}$, be a functions such that, f is differentiable and also alpha-differentiable. Let g be a function defined in the range of f and also differentiable, then we have the following rule:

$$D_x^\beta(g \circ f(x)) = \big(x + \frac{1}{\Gamma(\beta)}\big)^{1-\beta} f'(x) g'(f(x))$$

Definition 2.3. The sinc function is defined on the whole real line $-\infty < x < \infty$ by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Definition 2.4. For $h > 0$ and $k = 0, \pm 1, \pm 2, ...$ the translated sinc functions with space node are given by:

$$S(k, h)(x) = sinc\Big(\frac{x - kh}{h}\Big) = \begin{cases} \frac{\sin\left(\pi \frac{x-kh}{h}\right)}{\pi \frac{x-kh}{h}} & x \neq kh \\ 1 & x = kh. \end{cases}$$

Definition 2.5. If $f(x)$ is defined on the real line, then for $h > 0$ the series

$$C(f, h)(x) = \sum_{k=-\infty}^{\infty} f(kh) sinc\Big(\frac{x - kh}{h}\Big)$$

is called the Whittaker cardinal expansion of f whenever this series converges.

In general, approximations can be constructed for infinite, semi-infinite and finite intervals. To construct an approximation on the interval (a, b), the conformal map

$$\varphi(z) = \ln\Big(\frac{z - a}{b - z}\Big)$$

is employed. The basis functions on the interval (a, b) are derived from the composite translated sinc functions

$$S_k(z) = S(k, h)(z) \circ \varphi(z) = sinc\Big(\frac{\varphi(z) - kh}{h}\Big).$$

The inverse map of $w = \varphi(z)$ is

$$z = \varphi^{-1}(w) = \frac{a + be^w}{1 + e^w}.$$

The sinc grid points $z_k \in (a,b)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \varphi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \qquad k = 0, \pm 1, \pm 2, ...$$

Theorem 2.6. Let Γ be $(0,1), F \in B(D_E)$, then for $h > 0$ sufficiently small,

$$\int_{\Gamma} F(z)dz - h\sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\varphi'(z_j)} = \frac{i}{2}\int_{\partial D} \frac{F(z)k(\varphi,h)(z)}{\sin(\pi\varphi(z)/h)}dz \equiv I_F \tag{2.1}$$

where

$$|k(\varphi,h)|_{z\in\partial D} = \left| e^{\left[\frac{i\pi\varphi(z)}{h} sgn(Im\varphi(z))\right]} \right|_{z\in\partial D} = e^{\frac{-\pi d}{h}}.$$

For the sinc-Galerkin method, the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

Theorem 2.7. If there exist positive constants α, β and C such that

$$\left|\frac{F(x)}{\varphi'(x)}\right| \leq C \begin{cases} e^{-\alpha|\varphi(x)|} & x \in \psi((-\infty,\infty)) \\ e^{-\beta|\varphi(x)|} & x \in \psi((0,\infty)). \end{cases} \tag{2.2}$$

then the error bound for the quadrature rule (2.1) is

$$\left|\int_{\Gamma} F(x)dx - h\sum_{j=-M}^{N} \frac{F(x_j)}{\varphi'(x_j)}\right| \leq C\left(\frac{e^{-\alpha Mh}}{\alpha} + \frac{e^{-\beta Nh}}{\beta}\right) + |I_F| \tag{2.3}$$

The infinite sum in (2.1) is truncated with the use of (2.2) to arrive at the inequality (2.3). Making the selections

$$h = \sqrt{\frac{\pi d}{\alpha M}}$$

and

$$N \equiv \left[\left\lfloor \frac{\alpha M}{\beta} + 1 \right\rfloor\right]$$

where $[\lfloor . \rfloor]$ is the integer part of the statement and M is the integer value which specifies the grid size, then

$$\int_{\Gamma} F(x)dx = h\sum_{j=-M}^{N} \frac{F(x_j)}{\varphi'(x_j)} + O\left(e^{-(\pi\alpha dM)^{1/2}}\right). \tag{2.4}$$

These theorems are used for the integrals in the inner products that arise from the method presented here.

3 The sinc-Galerkin method

An approximate solution of $y(x)$ in (1.1) is represented by the formula

$$y_n(x) = \sum_{k=-M}^{N} c_k S_k(x), \qquad n = M + N + 1 \tag{3.1}$$

where S_k is function $S(k,h) \circ \varphi(x)$ for some fixed step size h. The unknown coefficients c_k in (3.1) are determined by orthogonalization of the residual with respect to the basis functions, i.e.

$$\langle y'', S_k \rangle + \langle p(x)y', S_k \rangle + \langle q(x)y, S_k \rangle + \langle n(x)y^s y^{(\beta)}, S_k \rangle = \langle f(x), S_k \rangle, \qquad k = -M, \dots, N \tag{3.2}$$

The inner product used for the sinc-Galerkin method is defined by

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx$$

where $w(x)$ a weight function which is taken for second-order boundary value problems in the following form

$$w(x) = \frac{1}{\varphi'(x)}.$$

We need the following theorems for the approximation of inner products in (3.2).

Theorem 3.1. The following relations hold:

$$\langle y'', S_k \rangle \approx h \sum_{j=-M}^{N} \sum_{i=0}^{2} \frac{y(x_j)}{\varphi'(x_j)h^i} \delta_{kj}^{(i)} g_{2,i}(x_j) \tag{3.3}$$

$$\langle p(x)y', S_k \rangle \approx -h \sum_{j=-M}^{N} \sum_{i=0}^{1} \frac{y(x_j)}{\varphi'(x_j)h^i} \delta_{kj}^{(i)} g_{1,i}(x_j) \tag{3.4}$$

and for $G(x) = q(x)y(x)$ and $G(x) = f(x)$

$$\langle G, S_k \rangle \approx h \frac{G(x_k)w(x_k)}{\varphi'(x_k)} \tag{3.5}$$

The proof of this theorem and values of $g_{k,i}(x)$ can be found in [14].

Theorem 3.2. The following relation holds:

$$\langle n(x)y^s y^{(\beta)}, S_k \rangle \approx$$
$$-\frac{h}{s+1} \sum_{j=-M}^{N} \frac{y^{s+1}(x_j)}{\varphi'(x_j)} \Big[\frac{1}{h} \delta_{kj}^{(1)} (\varphi' n w(x + \frac{1}{\Gamma(\beta)})^{1-\beta})(x_j) + \delta_{kj}^{(0)} (n w(x + \frac{1}{\Gamma(\beta)})^{1-\beta})'(x_j) \Big] \tag{3.6}$$

Proof. For $n(x)y^s y^{(\beta)}$, the inner product with sinc basis elements is given by

$$\langle n(x)y^s y^{(\beta)}, S_k\rangle = \int_a^b y^s (x + \frac{1}{\Gamma(\beta)})^{1-\beta}) y'(S_k n w) dx.$$

Integrating by parts to remove the first derivative from the dependent variable y leads to the equality

$$\langle n(x)y^s y^{(\beta)}, S_k\rangle = B - \frac{1}{s+1}\int_a^b y^{s+1}(S_k n w(x + \frac{1}{\Gamma(\beta)})^{1-\beta})' dx, \tag{3.7}$$

where the boundary term is

$$B = \Big[\frac{1}{s+1}(y^{s+1} S_k n w(x + \frac{1}{\Gamma(\beta)})^{1-\beta})\Big]_{x=a}^{b} = 0,$$

and expanding the derivatives under the integral in (3.7) yields

$$\langle n(x)y^s y^{(\beta)}, S_k\rangle = -\frac{1}{s+1}\int_a^b y^{s+1}\Big[S_k^{(1)}\varphi'(nw(x + \frac{1}{\Gamma(\beta)})^{1-\beta}) + S_k^{(0)}(nw(x + \frac{1}{\Gamma(\beta)})^{1-\beta})'\Big] dx. \tag{3.8}$$

Applying the sinc quadrature rule given by (2.4) to the right-hand side of (3.8) and deleting the error term yields (3.6). Q.E.D.

Replacing each term of (3.2) with the approximations defined in (3.3)-(3.6), respectively, and replacing $y(x_j)$ by c_j, and dividing by h, we obtain the following theorem:

Theorem 3.3. If the assumed approximate solution of the boundary-value problem (1.1) is (3.1), then the discrete sinc-Galerkin system for the determination of the unknown coefficients $\{c_j\}_{j=-M}^{N}$ is given, for $k = -M, \ldots, N$, by

$$\begin{aligned}\sum_{j=-M}^{N}\Bigg\{&\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2,i}(x_j)}{\varphi'(x_j)}c_j - \sum_{i=0}^{1}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{1,i}(x_j)}{\varphi'(x_j)}c_j - \frac{1}{s+1}\Big[\frac{1}{h}\delta_{kj}^{(1)}(nw(x + \frac{1}{\Gamma(\beta)})^{1-\beta})(x_j)c_j^{s+1}\\ &+ \frac{(nw(x+\frac{1}{\Gamma(\beta)})^{1-\beta})'(x_k)}{\varphi'(x_k)}c_k^{s+1}\Big]\Bigg\} + \frac{\mu_0(x_k)w(x_k)}{\varphi'(x_k)}c_k = \frac{f(x_k)w(x_k)}{\varphi'(x_k)}\end{aligned} \tag{3.9}$$

Now we define some notation to represent the system (3.9) in matrix-vector form. Let $\mathbf{D}(y)$ denote a diagonal matrix whose diagonal elements are $y(x_{-M}), y(x_{-M+1}), ..., y(x_N)$ and non-diagonal elements are zero; also for $0 \le i \le 2$, let $\mathbf{I}^{(i)}$ denote the matrices

$$\mathbf{I}^{(i)} = [\delta_{jk}^{(i)}], \qquad j,k = -M,\ldots,N.$$

where $\mathbf{I}$ and $\mathbf{D}$ are square matrices of dimension $n \times n$. In order to calculate the unknown coefficients c_k in the nonlinear system (3.9), we rewrite this system using the above notations in matrix-vector form as

$$\mathbf{AC} + \mathbf{BC}^s = \mathbf{F} \tag{3.10}$$

where

$$\mathbf{A} = \sum_{j=0}^{2} \frac{1}{h^j}\mathbf{I}^{(j)}\mathbf{D}\left(\frac{g_{2,j}}{\varphi'}\right) - \sum_{j=0}^{1} \frac{1}{h^j}\mathbf{I}^{(j)}\mathbf{D}\left(\frac{g_{1,j}}{\varphi'}\right) + \mathbf{I}^{(0)}\mathbf{D}\left(\frac{g_{0,0}}{\varphi'}\right)$$

$$\mathbf{B} = -\frac{1}{s+1}\left[\frac{1}{h}\mathbf{I}^{(1)}\mathbf{D}\Big(nw(x+\frac{1}{\Gamma(\beta)})^{1-\beta}\Big) + \mathbf{I}^{(0)}\mathbf{D}\left(\frac{(nw(x+\frac{1}{\Gamma(\beta)})^{1-\beta})'}{\varphi'}\right)\right]$$

$$\mathbf{F} = \mathbf{D}\left(\frac{wf}{\varphi'}\right)\mathbf{1}$$

$$\mathbf{C}^j = \left(c^j_{-M}, c^j_{-M+1}, \ldots, c^j_{N-1}, c^j_N\right)^T, \quad j = 1, s.$$

Now we have a nonlinear system of n equations in n unknown coefficients given by (3.10). Solving by *Newton's method*, we can obtain the unknown coefficients c_k that are necessary for approximating the solution in (3.1).

4 Error estimation

In this section, we define two error functions that are actual error function $E_N(x)$ and estimate error function $\tilde{E}_N(x)$ to check the accuracy of the presented method. Actual error function $E_N(x)$ is used in the problems that are known the exact solutions and it is defined by

$$E_N(x) = |y_N(x) - y(x)| \tag{4.1}$$

where $y(x)$ is the exact solution of Equation (1.1). The estimate error function $\tilde{E}_N(x)$ might be used in the problems that are unknown the exact solutions. If the $y_N(x)$ is an approximate solution to Equation (1.1), then when these functions and their derivatives are substituted into Equation (1.1) the obtained equation should be satisfied approximately. In short, for $x_k \in [a, b]$, the function $\tilde{E}_N(x)$ is defined by

$$\tilde{E}_N(x_k) = |L[y_N(x_k)] - f(x_k)| \cong 0 \tag{4.2}$$

and $\tilde{E}_N \leq 10^{-t_k}$ (t_k any positive contant). If $max 10^{-t_k} = 10^{-t}$ is prescribed, the truncation limit N is increased until the difference $\tilde{E}_N(x_k)$ at each of the points becomes smaller than the prescribed 10^{-t} [13].

5 Computational examples

In this section, two numerical examples are presented to show the accuracy of the present method. In the first example, a problem that has the known exact solution is selected. On this problem, the present method is tested via the error functions given by (4.1) and (4.2). In the second example, a singular problem that has the known exact solution for integer order derivative case is considered. So, the example is only tested via the error function given by (4.2). In the both examples, $N = M$ and $h = \pi/\sqrt{2N}$ is taken and the obtained results are illustrated via tables and graphics at selected points in the interval (a, b).

Example 1. Consider the following nonlinear fractional boundary value problem

$$y''(x) + xy'(x) + x^3 y^4(x) y^{(0.5)}(x) = f(x) \tag{5.1}$$

Table 1: Numerical values of the error function E_N for *Example 1* for different values of N

x	E_{16}	E_{32}	E_{64}	E_{128}
0.2	1.963×10^{-5}	5.729×10^{-7}	3.480×10^{-9}	2.374×10^{-12}
0.4	3.890×10^{-5}	1.124×10^{-6}	6.826×10^{-9}	4.656×10^{-12}
0.6	5.627×10^{-5}	1.632×10^{-6}	9.909×10^{-9}	6.764×10^{-12}
0.8	7.193×10^{-5}	2.082×10^{-6}	1.265×10^{-8}	8.634×10^{-12}

Table 2: Numerical values of the error function $\tilde{E}_N$ for *Example 1* for different values of N

x	$\tilde{E}_{16}$	$\tilde{E}_{32}$	$\tilde{E}_{64}$	$\tilde{E}_{128}$
0.2	1.603×10^{-4}	1.038×10^{-6}	6.317×10^{-10}	6.554×10^{-13}
0.4	2.095×10^{-4}	1.392×10^{-6}	5.639×10^{-9}	1.663×10^{-12}
0.6	1.581×10^{-4}	8.781×10^{-7}	5.740×10^{-9}	6.590×10^{-12}
0.8	3.614×10^{-4}	3.815×10^{-6}	3.687×10^{-10}	6.227×10^{-12}

subject to the homogeneous boundary conditions

$$y(0) = 0, \qquad y(1) = 0$$

where

$$f(x) = x(6 - 12x - 3(x-1)x^2 - x^3 - (x-1)^4 x^{16}(x + 1/\Gamma(0.5))^{0.5}(4x-3)).$$

The exact solution of Eq.(5.1) is given by $y(x) = x^3(1-x)$. Table 1 presents numerical values of the actual error functions obtained for Eq.(5.1) when $N = 16, 32, 64$ and 128. Similarly, in Table 2, numerical values of the estimated error functions obtained for Eq.(5.1) when $N = 16, 32, 64$ and 128 are presented. In Figure 1, it is given that graphics of exact solution and approximate solution obtained by the present method in the interval $(0,1)$ for $N = 4, 16$ and 64. In the interval $(0.2, 0.8)$, Figure 2 presents graphics of the estimated error functions for $N = 4, 16$ and 64. The presented tables and figures show that the approximate solutions converge the exact ones when it is increased that the number of sinc grid points N.

Example 2. Consider the nonlinear singular boundary value problem

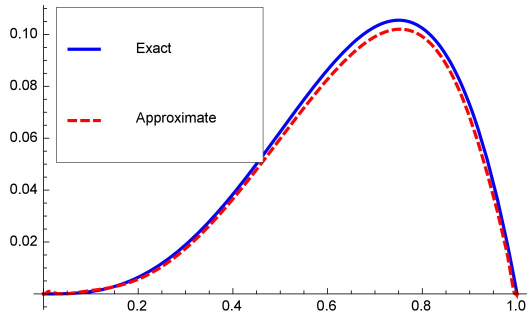
$$y''(x) - \frac{1}{x}y'(x) + \frac{1}{x(x-1)}y^3(x)y^{(\beta)}(x) - \frac{1}{x-1}y(x) = f(x) \tag{5.2}$$

subject to the homogeneous boundary conditions

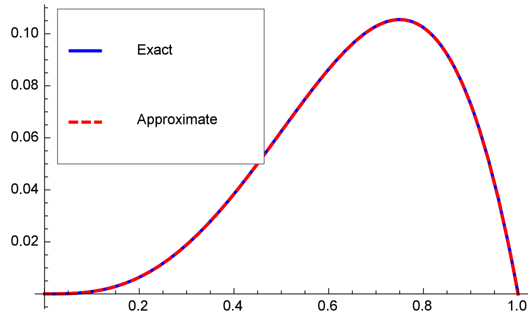
$$y(0) = 0, \qquad y(1) = 0$$

where

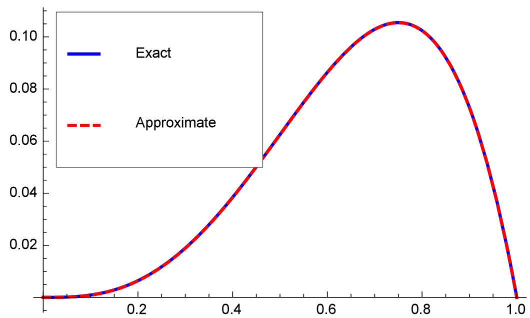
$$f(x) = \pi\cos(\pi x) - \frac{\sin(\pi x)}{x} - \pi^2 x\sin(\pi x) - \frac{x(\sin\pi x)}{x-1} + \frac{\pi x^3\cos(\pi x)(\sin(\pi x))^3}{x-1} + \frac{x^2(\sin(\pi x))^4}{x-1}.$$



(a) $N = 4$

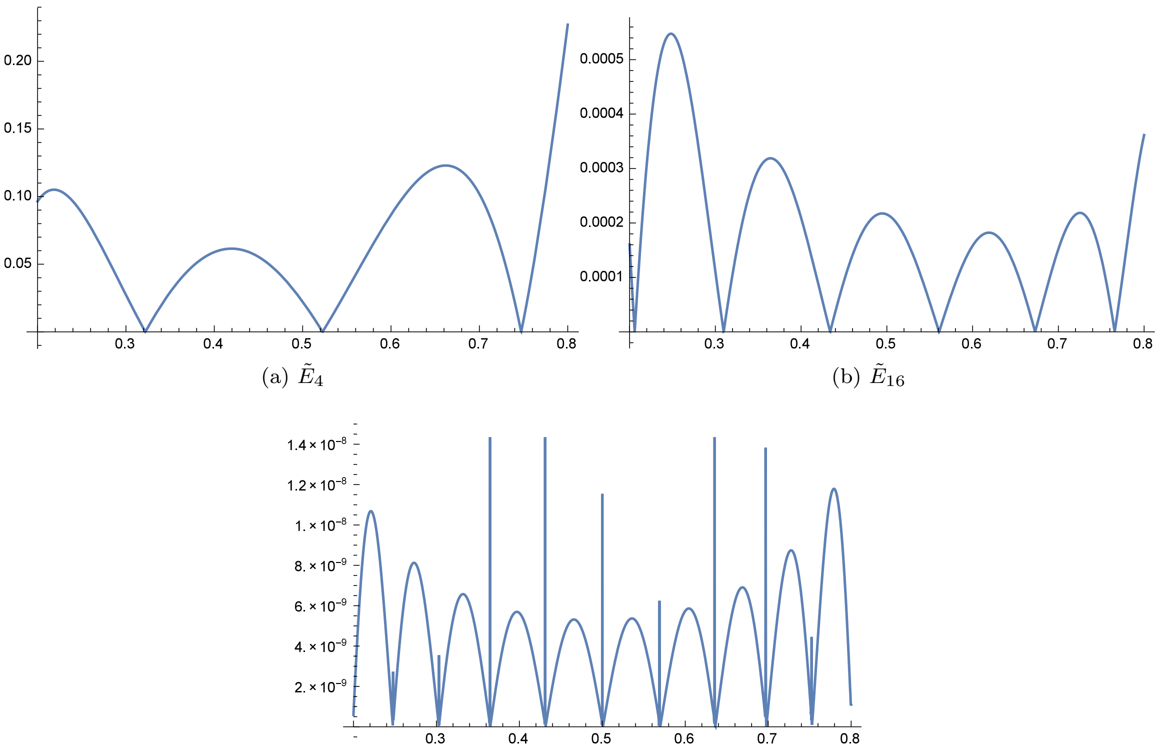


(b) $N = 16$



(c) $N = 64$

Figure 1: Graphics of exact and approximate solutions for Example 1 when $N = 4, 16, 64$.



(a) $\tilde{E}_4$

(b) $\tilde{E}_{16}$

(c) $\tilde{E}_{64}$

Figure 2: Graphics of the error function $\tilde{E}_N$ for Example 1 when $N = 4, 16, 64$.

Table 3: Numerical results for different values of β when $N = 128$ for *Example 2*

x	$\beta = 0.1$	$\beta = 0.5$	$\beta = 0.9$	$\beta = 1$
0.2	0.119230	0.117859	0.117525	0.117557
0.4	0.386105	0.381504	0.380330	0.380423
0.6	0.576963	0.573097	0.570897	0.570634
0.8	0.473469	0.474663	0.471291	0.470228

Table 4: Numerical values of $\tilde{E}_N$ for different values of N when $\beta = 0.5$ in *Example 2*

x	$\tilde{E}_{16}$	$\tilde{E}_{32}$	$\tilde{E}_{64}$	$\tilde{E}_{128}$
0.2	2.076×10^{-2}	4.206×10^{-4}	2.118×10^{-6}	2.841×10^{-10}
0.4	1.457×10^{-2}	2.468×10^{-4}	4.545×10^{-7}	1.065×10^{-10}
0.6	1.083×10^{-2}	1.697×10^{-4}	9.739×10^{-7}	1.622×10^{-10}
0.8	1.434×10^{-2}	6.990×10^{-4}	3.477×10^{-7}	2.137×10^{-10}

The exact solution of Eq.(5.2) for $\beta = 1$ is $y(x) = x \sin \pi x$. Table 3 presents numerical values obtained by different values of β for Eq.(5.2) when $N = 128$. Similarly, in Table 4, numerical results of the estimated error functions obtained for $\beta = 0.5$ when $N = 16, 32, 64$ and 128 are presented. In Figure 3, it is given that graphics of approximate solutions obtained by the present method for $\beta = 0.1, 0.5, 0.9, 1$ in the interval $(0.4, 0.41)$ when $N = 128$. In the interval $(0.2, 0.8)$, Figure 4 presents graphics of the estimated error functions for $N = 4, 16$ and 64. The presented tables and figures show that the approximate solutions converge the exact ones when it is increased that the number of sinc grid points N.

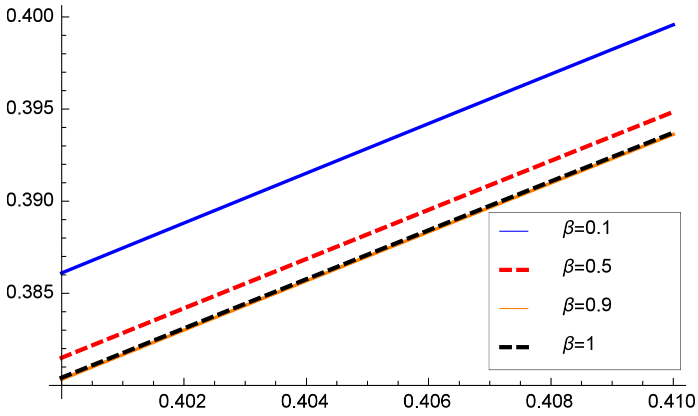


Figure 3: The zoomed graphics of approximate solutions for different values of β in Example 2 when $N = 128$

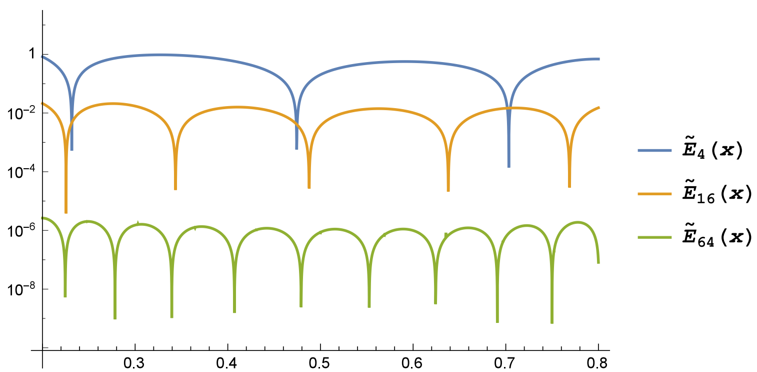


Figure 4: The graphics of the estimated error functions $\tilde{E}_N$ for $\beta = 0.3$ in Example 2 when $N = 4, 16, 64$

6 Conclusion

In this research, sinc-Galerkin method is applied to obtain the approximate solutions of a class of fractional nonlinear differential equations (1.1). In order to illustrate the accuracy of the presented method, the numerical results obtained from various levels of iterations are compared with exact solutions. As a result of those comparisons, it is clear that sinc-Galerkin method provides a good approximation and shows promise for solving different types of fractional nonlinear differential equations. In addition to the results of comparison, the error estimation guarantees the efficiency of the proposed method.

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